Interactions

- We can couple a scalar field φ to the Dirac equation by introducing the term
$$g\varphi\bar{\psi}\psi$$
in the Lagrangian

- Similarly, we can couple a vector field by adding
$$eA_\mu\bar{\psi}\gamma^\mu\psi$$

$\rightarrow$ introduce covariant derivative
$$D_\mu := \partial_\mu - ieA_\mu$$
and write
$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi$$
$$= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$\rightarrow$ the total Lagrangian for a Dirac field interacting with vector field of mass μ:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu \tag{1}$$

If μ vanishes, eq. (1) is invariant under gauge transformations

$$A_\mu(x) \longmapsto A_\mu(x) + \partial_\mu \varepsilon(x)$$

$$\delta \psi(x) = i\varepsilon(x)\, e\, \psi(x)$$

$$\rightarrow \quad \delta S = -\int d^4x\, j^\mu(x)\, \partial_\mu \varepsilon(x) \quad \text{(Noether)}$$

with $\quad j^\mu = -i \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\, e\psi \quad$ and $\partial_\mu j^\mu = 0$

$\longrightarrow$ A_μ couples to the conserved current

$$j^\mu = \bar{\psi}\gamma^\mu \psi$$

Varying (1) ($\mu = 0$) with respect to $\bar{\psi}$, we obtain :

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi = 0 \qquad (2)$$

<u>Charge conjugation and anti-matter</u>

With coupling to the electromagnetic field, we have arrived at the concept of charge

$\longrightarrow$ consider $[-i\gamma^{\mu *}(\partial + ieA_\mu) - m]\psi^* = 0 \quad (3)$

(complex conj. of (2))

$-\gamma^{\mu *}$ also satisfy the Clifford alg.

$$\left(\{\gamma^{\mu *}, \gamma^{\nu *}\} = 2\eta^{\mu\nu} \right)$$

→ γ- matrices just in different basis

thus there must be a matrix C, such that

$$-\gamma^{\mu *} = (C\gamma^0)^{-1}\gamma^\mu (C\gamma^0) \qquad (4)$$

<u>Note</u>: the choice $C\gamma^0$ is conventional

Plugging (4) back to (3), we find

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi_c = 0 \qquad (5)$$

where $\psi_c := C\gamma^0\psi^*$

→ the field ψ_c described by (5), has same mass but opposite charge (namely $e' = -e$) as compared to (2) !

→ have found the "positron" (anti-matter)

We write the defining eq. for C as

$$C\gamma^0\gamma^{\mu *}\gamma^0 C^{-1} = -\gamma^\mu$$

complex conjugating $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$, we get

$$(\gamma^\mu)^T = \gamma^0 \gamma^{\mu *} \gamma^0 \quad \text{if } \gamma^0 \text{ real}$$

$\longrightarrow$ $$(\gamma^\mu)^T = -C^{-1} \gamma^\mu C$$

In both Dirac and Wey basis γ^2 is only immaginary γ-matrix

$\longrightarrow$ equation (4) says that $C\gamma^0$ commutes with γ^2 but anticommutes with $\gamma^1, \gamma^0, \gamma^3$

$\longrightarrow$ $C = \gamma^2 \gamma^0$ (up to phase)

you can check: $\gamma^2 \gamma^{\mu *} \gamma^2 = \gamma^\mu$

$\longrightarrow$ $\psi_c = \gamma^2 \psi^*$

Can also check:

- $\gamma_5 (\psi_L)_c = + (\psi_L)_c \rightarrow$ right-handed
- $\gamma_5 (\psi_R)_c = - (\psi_R)_c \rightarrow$ left-handed

$\longrightarrow$ charge conjugation reverses chirality!

Under Lorentz trfs. ψ_c transforms as

$$\psi \mapsto e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi$$

$$\rightarrow \quad \psi^* \mapsto e^{+\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^*}\psi^*$$

hence $\psi_c \mapsto \gamma^2 e^{+\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^*}\psi^*$

$$= e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi_c$$

(recall that $\sigma^{\mu\nu}$ is defined with explicit i)

<u>Note</u>: $C^T = \gamma^0\gamma^2 = -C$ in both Dirac and Weyl bases

<u>Majorana fermions</u>

Since ψ_c transforms as a spinor, Lorentz invariance allows the equation:

$$i\partial\!\!\!/\,\psi = m\psi_c \qquad (6)$$

$\rightarrow$ complex conjugating and multiplying by γ^2, gives

$$-\gamma^2 i\gamma^{\mu *}\partial_\mu\psi^* = \gamma^2 m(-\gamma^2)\psi$$

$\rightarrow \; i\cancel{\partial}\psi_c = m\psi$

and thus

$$-\partial^2\psi = i\cancel{\partial}(i\cancel{\partial}\psi)$$

$$\left(= -\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu\psi \right.$$

$$\left. = -\eta^{\mu\nu}\partial_\mu\partial_\nu\psi \right)$$

$$= i\cancel{\partial} m\psi_c$$

$$= m^2\psi$$

$\rightarrow$ m is indeed the mass of the particle associated to ψ

The Majorana eq. (6) can be obtained from the Lagrangian

$$\mathcal{L} = \bar{\psi} i\cancel{\partial}\psi - \frac{1}{2}m(\psi^T C\psi + \bar{\psi}C\bar{\psi}^T)$$

upon varying $\bar{\psi}$.

<u>Note:</u> The Majorana eq. preserves handedness (contrary to Dirac eq.). This can be checked by multiplying eq. (6) from both sides with γ_5 and check that the eigenvalues match

Time reversal:

In 1932 Wigner showed that time reversal is represented by anti-unitary operator:

Schrödinger eq. $i \frac{\partial}{\partial t} \psi(t) = H \psi(t)$

(say $H = -\frac{1}{2m} \nabla^2 + V(\vec{x})$)

→ consider transformation $t \mapsto t' = -t$

want to find $\psi'(t')$ such that

$i \left(\frac{\partial}{\partial t'}\right) \psi'(t') = H \psi'(t')$

write $\psi'(t') = T\psi(t)$, where T some operator

$\rightarrow i \frac{\partial}{\partial(-t)} T\psi(t) = HT\psi(t) \quad | \cdot T^{-1}$

$T^{-1}(-i) T \frac{\partial}{\partial t} \psi(t) = T^{-1} H T \psi(t) = H \psi(t)$

(since H is indep. of t)

$\rightarrow T^{-1}(-i) T = i$

"In quantum physics, flipping time means flipping i as well."

Let now $T = UK$, where K is compl. conj.

$\rightarrow \; T^{-1} = K U^{-1} \;\longrightarrow\; U^{-1} i U = i$

"anti-unitary"

For a spinless particle in a plane wave state $\psi(t) = e^{i(\vec{k}\cdot\vec{x} - Et)}$, we get

$$\psi'(t') = T\psi(t) = UK\psi(t) = U\psi^*(t)$$
$$= U e^{-i(\vec{k}\cdot\vec{x} - Et)}$$

U is just a phase here $\rightarrow$ set $U = 1$

$$\rightarrow \psi'(t) = e^{-i(\vec{k}\cdot\vec{x} + Et)} = e^{i(-\vec{k}\cdot\vec{x} - Et)}$$

thus ψ' describes a plane wave moving in the opposite direction

Note: acting on a spinless particle, we get $T^2 = UKUK = UU^*K^2 = +1$

Let's now look at the Dirac equation

Multiplying

$$[i\not{\partial} - m]\psi = 0$$

from the left with γ^0 gives

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \qquad H = -i\gamma^0\gamma^i\partial_i + \gamma^0 m$$

Once again, we want

$$i\left(\frac{\partial}{\partial t'}\right)\psi'(t') = H\psi'(t'), \quad \psi'(t') = T\psi(t)$$

impose $T^{-1}HT = H$, set $T = UK$

complex conjugation

$$\rightarrow \quad KU^{-1}HUK = H$$

giving

$$KU^{-1}\gamma^0 UK = \gamma^0, \quad KU^{-1}(i\gamma^0\gamma^i)UK = i\gamma^0\gamma^i$$

$\rightarrow$ have to solve for a U s.t.

$$U^{-1}\gamma^0 U = \gamma^{0*} \quad \text{and} \quad U^{-1}\gamma^i U = -\gamma^{i*}$$

Next, restrict to Dirac and Weyl basis

$\rightarrow$ γ^2 is only imaginary matrix

(doesn't change sign under $-\gamma^{2*}$)

thus we see

$$U = \eta\gamma^1\gamma^3$$

does the job! $\rightarrow \psi'(t') = \eta\gamma^1\gamma^3 K\psi(t)$

We have

$$U = \eta(\sigma^1 \otimes i\sigma_2)(\sigma^3 \otimes i\sigma_2) = \eta i\sigma^2 \otimes \mathbb{1}$$

<u>Note:</u> $T^2\psi = -\psi$

CPT theorem:

Any local Lorentz invariant field theory must be invariant under combined time-reversal, Parity, and charge conjugation operations:

"CPT invariance"

We have

$$CPT\,\phi(x)\,[CPT]^{-1} = \zeta^*\xi^*\eta^*\phi^\dagger(-x)$$

$$CPT\,\phi_\mu(x)\,[CPT]^{-1} = -\zeta^*\xi^*\eta^*\phi_\mu^\dagger(-x)$$

$$CPT\,\psi(x)\,[CPT]^{-1} = -\zeta^*\xi^*\eta^*\gamma_5\,\psi^*(-x)$$

$\longrightarrow$ choose $\zeta\xi\eta = 1$

Can check that any interaction term built from these is invariant under CPT